STRONG EXCEPTIONAL SEQUENCES OF VECTOR BUNDLES ON CERTAIN FANO VARIETIES

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Abstract. Exceptional sequences of vector bundles/sheaves over a variety X are special generators of the triangulated category $D^b(Coh\,X)$. Kapranov proved the existence of tilting bundles over homogeneous varieties for the general linear group. King conjectured in [7] the existence of tilting sequences of vector bundles on projective varieties which are obtained as quotients of Zariski open subsets of affine spaces.

Although the conjecture does not hold in general, it remains the problem of constructing examples of varieties admitting tilting bundles. For toric varieties, examples of exceptional bundles have been given by Altmann and Hille in [1], and by Costa and Miró-Roig in [3].

The goal of this paper is to give further examples of projective varieties carrying exceptional sequences of vector bundles. The varieties are obtained as geometric invariant quotients of affine spaces by linear actions of reductive groups, as in King's conjecture.

Introduction

The abstract concept of derived categories has been introduced by Grothendieck and further developed by Verdier. However, their work remained within a very general and abstract setting, and one wished to have concrete examples arising from geometry. In algebraic geometry one of the essential objects associated to a projective variety is the (bounded) derived category of coherent sheaves over it. Its knowledge allows to recover all the cohomological data of the variety.

Beilinson made the first major step by proving that the line bundles

$$\mathcal{O}_{\mathbb{P}^n}(-n), \ldots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}$$

generate $D^b(\mathrm{Coh}\,\mathbb{P}^n)$, and actually form a strong, complete, exceptional sequence. Afterwards have appeared several other examples of varieties admitting (strong and complete) exceptional sequences of vector bundles. One of the most notable results in this direction is that of Kapranov [6].

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He explicitly constructed a strong exceptional sequence of vector bundles over homogeneous varieties for $\mathrm{Gl}(n)$ (that is over Grassmannians and flag manifolds). Further examples, which are based on Kapranov's result, have been obtained in [3].

In the unpublished preprint [7], King conjectured that there are exceptional sequences over projective varieties which are obtained as invariant quotients of affine spaces for the linear action of reductive groups. More precisely, let G be a complex reductive group and let V be a finite dimensional G-module. If χ is a character of G, then the geometric invariant theory delivers a G-invariant Zariski open subset $V^{\mathrm{ss}}(\chi) \subset V$, and a categorical quotient $Y := V^{\mathrm{ss}}(\chi)/G$. Observe that the flag varieties for $Gl(n, \mathbb{C})$, and toric varieties are special cases of such varieties Y.

We invite the reader to consult [2] for the terminology of exceptional sequences of sheaves over projective varieties, and for an up-to-date survey of their properties.

The answer to King's conjecture is negative in general. Hille and Perling gave in [5] an example of a toric variety ($\mathbb{P}^2$ blown-up repeatedly three times) with the property that it does not admit a tilting object formed by line bundles.

However it is still a very interesting problem to find classes of examples for which the conjecture is true. I wish to mention also the article [1] where the authors proved the existence of (partial) strong exceptional sequences on quotients arising from thin representations of quivers.

In this note I give further examples of strong exceptional sequences of vector bundles over certain Fano varieties. The varieties considered in this paper are obtained as geometric quotients of open subsets of affine spaces by the linear action of a reductive group. The main result of this paper is the following:

Theorem *Consider a faithful representation $\rho\colon G \to \mathrm{Gl}(V)$, and let $\mathbb{V}$ be the affine space corresponding to V. We assume that the ring of invariants $K[\mathbb{V}]^T = K$. We assume moreover that the characters of G on the various isotypical components of V are pairwise distinct.*

We denote $Y_m := \mathbb{V}^m(G, \chi_{\mathrm{ac}})/G$, and $\mathcal{M}_\omega \to Y_m$ the vector bundle induced by the isotypical component M_ω of V. Then $(\mathcal{M}_\omega)_\omega$ forms a strong exceptional sequence over Y_m for $m \gg 0$.

We should point out that these examples substantially extend those presented in [1], Theorem 3.6. The examples constructed in *loc. cit.* correspond to the case where V is a thin representation space of a quiver.

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1. A STABILITY PROPERTY

In the whole paper G denotes a connected, reductive group over an algebraically closed field K of characteristic zero, and T denote its maximal torus. We consider a faithful representation $\rho\colon G \to \mathrm{Gl}(V)$, and denote by $\mathbb{V} := \mathrm{Spec}(\mathrm{Sym}^\bullet V^\vee)$ the corresponding affine space. We assume that the invariant ring $K[\mathbb{V}]^T = K$; it follows automatically that $K[\mathbb{V}]^G = K$.

Lemma 1.1. *Let V be a non-zero G-module such that $K[\mathbb{V}]^T = K$. Then holds:*

(1) *There is a 1-PS $\lambda \in \mathcal{X}_*(T)$ such that all its weights on V are strictly positive.*
(2) *G is not semi-simple.*

Proof. (i) Let Φ denote the set of weights of the T-module V. Then the set of weights of the T on $K[\mathbb{V}]$ is the 'cone' $\sum\limits_{\eta\in\Phi} \mathbb{N}\eta$. Since $K[\mathbb{V}]^T = K$, this cone is strictly convex. Otherwise we could construct a non-trivial T-invariant monomial. It follows that there is $\lambda \in \mathcal{X}_*(T)$ with $\langle \eta, \lambda\rangle > 0$ for all $\eta \in \Phi$.

(ii) Assume that G is semi-simple. The previous step implies that $K[\mathbb{V}^m]^T = K$, and therefore $K[\mathbb{V}^m]^G = K$ for all $m \geqslant 1$. Since G is semi-simple, it follows that G has an open orbit in $\mathbb{V}^m$. For large m we get a contradiction. □

Let us decompose now $V = \bigoplus\limits_{\omega\in\mathcal{X}} M_\omega^{\oplus\nu_\omega}$ into its G-isotypical components. Then $Z(G)^\circ$ acts on M_ω by a character which we denote χ_ω. We let $d_\omega := \dim M_\omega$.

Lemma 1.2. *Assume that $\nu_\omega \geqslant d_\omega$. Then the character χ_ω is effective (that is, the semi-stable locus $\mathbb{V}(G,\chi_\omega)$ is not empty).*

Proof. We can re-write $V = \bigoplus_{\omega\in\mathcal{X}} \mathrm{Hom}(K^{\nu_\omega}, M_\omega)$. Since $\nu_\omega \geqslant d_\omega$, it makes sense to associate to an element $\mathrm{Hom}(K^{\nu_\omega}, M_\omega)$ the $d_\omega \times d_\omega$-minor corresponding to the first d_ω columns. This defines a regular function f_ω which is $d_\omega\chi_\omega$-equivariant; moreover, f_ω does not vanish on surjective homomorphisms. It follows that $d_\omega\chi_\omega$, and therefore χ_ω, is effective for all ω. □

We start by proving a general stability result which is of independent interest. It is well known that the tangent bundle of the projective space is stable, or more generally that the tautological bundles over Grassmannians are stable. The goal of this section is to present an unified approach, and also to generalize these results.

We denote $\{G_j\}_{j\in J}$ the simple factors of G, and for each j we denote $\gamma_j\colon G \to G_j$ be the corresponding quotient morphism. Using the γ_j's we extend the structural group of $\Omega \to Y$, and obtain the principal G_j-bundles $\Omega(G_j) \to Y$. The main result of this section is:

Theorem 1.3. *Assume $\chi \in \mathcal{X}^*(G)$ has the property that G acts freely on $\Omega := \mathbb{V}^{\rm ss}(G,\chi)$, and let Y be the quotient. Assume that $\nu_\omega \geqslant \dim M_\omega$ holds for all $\omega \in \mathcal{X}$. Then the principal G_j-bundles $\Omega(G_j) \to Y$, $j \in J$, obtained by extending the structural group are semi-stable.*

Proof. We fix $j \in J$, and a maximal parabolic subgroup $P_j \subset G_j$; we let $P := \gamma_j^{-1}P_j$: it is a maximal parabolic subgroup of G. Observe that in this case the associated homogeneous bundles $\big(\Omega(G_j)\big)(G_j/P_j)$ and $\Omega\big(G/P\big)$ are isomorphic.

We denote $H = \prod\limits_{\omega} H_\omega := \prod\limits_{\omega} \mathrm{Gl}_K(\nu_\omega)$: it acts naturally on $\mathbb{V}$, and the G- and H-actions on $\mathbb{V}$ commute. It follows that H still acts on $\Omega(G/P)$ by

$$H \times \Omega(G/P) \longrightarrow \Omega(G/P), \quad h \times [y, gP] := [hy, gP].$$

We must prove that for any reduction of the structural group

$$s\colon Y^o \to \big(\Omega(G_j)\big)(G_j/P_j) = \Omega\big(G/P\big), \qquad \text{with } Y^o \subset Y \text{ open and } \mathrm{codim}_Y(Y\setminus Y^o) \geqslant 2,$$

we have $\deg_Y\big(s^*\mathsf{T}_{\Omega(G/P)/Y}\big) \geqslant 0$. Equivalently, one can view the reduction s as a G-equivariant morphism $S\colon \Omega^o = q^{-1}(Y^o) \to G/P$.

The idea is to move s using the H-action on $\Omega(G/P)$. Let $\hat{y} \in Y$ be a generic point, and consider $y \in \Omega$ over $\hat{y}$. We define the following subgroups of H:

$$\begin{aligned} H_{\hat{y}} &:= \{h \in H \mid \exists\, g_h \in G \text{ s.t. } hy = \rho(g_h^{-1})y\} \\ &= \prod_{\omega} H_{\omega,\hat{y}}, \quad \text{and} \quad K_{\hat{y}} := \mathrm{Stab}_H(y). \end{aligned}$$

We observe that $K_{\hat{y}}$ does not depend on the choice of $y \in q^{-1}(\hat{y})$. Since G acts freely on Ω, the assignment $h \mapsto g_h$ defines a group homomorphism $\rho_{\hat{y}}\colon H_{\hat{y}} \to G$ whose kernel is $K_{\hat{y}}$. We move the section s using the action of $H_{\hat{y}}$. For $h \in H_{\hat{y}}$ define a new section s_h as follows:

$$s_h(\hat{x}) := [x, S(h^{-1}\times x)] \qquad \text{(Equivalently, } S_h(x) := S(h^{-1}\times x) \text{ is } G\text{-equivariant).}$$

Observe that as $h \in H_{\hat{y}}$ varies, $s_h(\hat{y}) = h \times s(\hat{y})$ moves in the vertical direction.
Claim $H_{\hat{y}}/K_{\hat{y}} \to G/Z(G)^\circ$ is surjective. Write $y = (y_\omega)_\omega$ with respect to the direct sum decomposition of V; for each $\omega \in \mathcal{X}$, $y_\omega = (y_{\omega 1}, \dots, y_{\omega \nu_\omega})$. Since $y \in \Omega$ is chosen generically, and $\nu_\omega \geqslant \dim M_\omega =: d_\omega$, we may assume that for each $\omega \in \mathcal{X}$ the vectors $y_{\omega 1}, \dots, y_{\omega \nu_\omega}$ span M_ω. Equivalently, we may view y_ω as a surjective homomorphism $K^{\nu_\omega} \to M_\omega$.

For $g \in G$ holds $\rho(g)y = (\rho_\omega(g)y_\omega)_\omega$. Using that $\nu_\omega \geqslant d_\omega$, we deduce that for each $\omega \in \mathcal{X}$ there is $h_\omega \in \mathrm{Gl}_K(\nu_\omega)$ such that $h_\omega y_\omega = \rho_\omega(g^{-1})y_\omega$. For $h := (h_\omega)_\omega$ we have $hy = \rho(g^{-1})y$, that is $g \in \mathrm{Image}\big(H_{\hat{y}}/K_{\hat{y}} \to G\big)$. This proves our claim.

Back to our proof. We observe that the infinitesimal action of $H_{\hat{y}}$ preserves the restriction to the fibre $q^{-1}(\hat{y}) = \{[y, gP] \mid g \in G\} \cong G/P$ of the relative tangent bundle $\mathsf{T}_{\Omega(G/P)/Y}$. By this isomorphism the relative tangent bundle corresponds to $\mathsf{T}_{G/P} \to G/P$.

The previous claim implies that the infinitesimal action $\mathcal{L}ie(H_{\hat{y}}) \to \mathsf{T}_{\Omega(G/P)/Y, s(\hat{y})}$ is surjective. Therefore we can produce a section $\sigma \in H^0(Y^o, s^*\mathrm{det}\mathsf{T}_{\Omega(G/P)/Y})$ which does not vanish at the point $\hat{y}$. This proves that $\deg_Y\big(s^*\mathsf{T}_{\Omega(G/P)/Y}\big) \geqslant 0$. □

Corollary 1.4. *Assume $\chi \in \mathcal{X}^*(G)$ has the property that G acts freely on $\Omega := \mathbb{V}^{\mathrm{ss}}(G, \chi)$, and let Y be the quotient. For $\omega \in \mathcal{X}$ we denote by $\mathcal{M}_\omega := \Omega(M_\omega)$ the associated vector bundle over Y. Assume that $\nu_\omega \geqslant \dim M_\omega$ holds for all $\omega \in \mathcal{X}$. Then the vector bundles $\mathcal{M}_\omega \to Y$, $\omega \in \mathcal{X}$, are slope semi-stable with respect to the polarization defined by the character χ.*

Proof. We may assume that $G = Z(G)^\circ \times \big(\underset{j \in J}{\times} G_j\big)$. Since each $\Omega(G_j)$ is semi-stable, $\Omega \to Y$ itself is semi-stable. The homomorphism $\rho_\omega \colon G \to \mathrm{Gl}(M_\omega)$ maps $Z(G)^\circ$ into the centre of $\mathrm{Gl}(M_\omega)$. Using [9], theorem 3.18, we deduce that the associated vector bundle $\mathcal{M}_\omega = \Omega(M_\omega) \to Y$ is semi-stable. □

Example 1.5. Using these ideas we construct a *directed system* of principal G-bundles $\Omega_m \to Y_m$ which approximates the classifying space $\mathrm{EG} \to \mathrm{BG}$ (for details consult [4]). The special feature of this system is that the G-bundles $\Omega_m \to Y_m$ are *semi-stable.*

Let G be a connected, reductive group which is not semi-simple; then $G = (Z \times G')/F$, where $Z := Z(G)^\circ$ is positive dimensional and F is a finite group. Consider moreover a faithful representation $\rho \colon G \to \mathrm{Gl}(V)$. Twisting ρ with a suitable 'large' character of Z, we may assume that $K[\mathbb{V}]^T = K$, and $\chi := \det(\rho)$ is not in a wall for the G-action on the $\mathbb{V}^m$'s, simultaneously for all m. Then $K[\mathbb{V}^m]^G = K$ for all $m \geqslant 1$, and for m sufficiently large the multiplicity condition in theorem 1.3 is fulfilled.

The inclusion $\mathbb{V}^m \hookrightarrow \mathbb{V}^{m+1}$, $x \mapsto (x, 0)$, restricts to $\Omega_m := (\mathbb{V}^m)^{\mathrm{ss}}(G, \chi) \hookrightarrow \Omega_{m+1} := (\mathbb{V}^{m+1})^{\mathrm{ss}}(G, \chi)$, which induces the morphisms $Y_m \to Y_{m+1}$ at the quotient level. Since the quotients $\Omega_m \to Y_m$ are geometric, $Y_m \to Y_{m+1}$ is a closed embedding.

Assume moreover that G acts freely on the Ω_m's, for m large enough. What we obtain is a directed system $\Omega_m \to Y_m$ of *semi-stable* principal G-bundles over projective bases. Since the codimension of the unstable locus $(\mathbb{V}^m)^{\mathrm{us}}(G, \chi) \hookrightarrow \mathbb{V}^m$ grows at least linearly with m, these principal bundles form algebro-geometric substitutes for $\mathrm{EG} \to \mathrm{BG}$.

We remark that in the case $G = \mathrm{Gl}(d)$, together with its standard representation $V = K^d$, this construction yields the Grassmannian $\mathrm{Grass}(m, d)$ endowed with the universal quotient bundle.

2. A numerical criterion

In this section we present a preparatory result needed for proving the proposition 3.3. For a G-module W, let $\eta_1, \ldots, \eta_R$ be the weights of the maximal torus $T \subset G$. We define:

$$m : W \times \mathcal{X}_*(G)_{\mathbb{R}} \longrightarrow \mathbb{R},$$

$$m(w, \lambda) := \min\{\, j \mid \text{the } t^j\text{th-isotypical component of } w \text{ w.r.t. } \lambda \text{ does not vanish}\}.$$

Observe that for $\lambda \in \mathcal{X}_*(T)$ holds:

$$m(w, \lambda) := \min\{\langle \eta_j, \lambda\rangle \mid \text{the } \eta_j\text{-isotypical component of } w \text{ does not vanish}\}.$$

Consider now a character $\theta \in \mathcal{X}^*(G)$. We have the following numerical criterion for the (G, θ)-(semi-)stability:

$$w \in W^{\mathrm{s}}(G, \theta) \ \big(\text{resp. } w \in W^{\mathrm{ss}}(G, \theta)\big) \ \Leftrightarrow \inf \left\{ \tfrac{\langle \theta, \lambda\rangle}{|\lambda|} \;\middle|\; m(w, \lambda) \geqslant 0 \right\} \underset{(\geqslant)}{>} 0$$
$$\Leftrightarrow \left[\; m(w, \lambda) \geqslant 0 \ \Rightarrow\ \langle \theta, \lambda\rangle \underset{(\geqslant)}{>} 0 \;\right].$$

For $w \in W$ we define the sets

$$\mathscr{C}_w^G := \{\lambda \in \mathcal{X}_*(G) \mid m(w, \lambda) \geqslant 0\} \ \text{ and } \ \mathscr{C}_w^T := \{\lambda \in \mathcal{X}_*(T) \mid m(w, \lambda) \geqslant 0\}.$$

We observe that $\mathscr{C}_w^T$ is a convex, polyhedral cone, and only *finitely many* cones $\mathscr{C}_w^T$ occur as w varies in $W^s(G, \theta)$. We denote them by $\mathscr{C}_1, \ldots, \mathscr{C}_z$.

Moreover,

$$\mathscr{C}_w^G = \bigcup_{g\in G} \mathrm{Ad}_{g^{-1}}\big(\mathscr{C}_{gw}^T\big).$$

Since θ is Ad_G-invariant, we can reformulate the numerical criterion as follows:

$$\theta \in \mathcal{X}^*(G) \cap \bigcap_{w\in W^{\mathrm{s}}(G,\theta)} \mathscr{C}_w^\vee = \mathcal{X}^*(G) \cap \mathscr{C}_1^\vee \cap \ldots \cap \mathscr{C}_z^\vee.$$

Let V, M be two G-modules. We denote $E := \mathrm{End}(M)$, and $r = \dim E$. We consider the $K^\times \times G$-module $W_m := E \times V^m$, with the module structure given by

$$(t,g) \times \big(\varphi, (v_j)_j\big) := \big(t\cdot \mathrm{Ad}_{\rho(g)}\varphi, (\rho(g)v_j)_j\big),$$

and also $\theta_m := (r+1)\chi_t + m\chi_{\mathrm{ac}}(V) \in \mathcal{X}^*(K^\times \times G)$. Observe that

$$\begin{aligned} &m(\underline{v},\lambda) = \min_j m(v_j,\lambda), \quad \forall\, \underline{v} = (v_j)_j \in V^m, \quad \forall\, \underline{v} \in V^m,\\ &m((\varphi,\underline{v}), t^\varepsilon\lambda) = \min\{\varepsilon + m(\varphi,\lambda), m(v_j,\lambda) \mid j = 1,\ldots,m\}. \end{aligned}$$

The numerical criterion for $E\times V^m$ reads: a point $w = (\varphi, \underline{v})$ is stable w.r.t. $(K^\times \times G, \theta_m)$ if and only if

(2.1)

$$\begin{cases} (A) & m(\varphi,\lambda) \geqslant 0, \quad m(\underline{v},\lambda) \geqslant 0 \quad \Rightarrow \quad \langle\chi_{\mathrm{ac}},\lambda\rangle > 0;\\ (B) & 1 + m(\varphi,\lambda) \geqslant 0, \quad m(\underline{v},\lambda) \geqslant 0 \quad \Rightarrow \quad (r+1) + m\cdot\langle\chi_{\mathrm{ac}},\lambda\rangle > 0;\\ (C) & -1 + m(\varphi,\lambda) \geqslant 0, \quad m(\underline{v},\lambda) \geqslant 0 \quad \Rightarrow \quad -(r+1) + m\cdot\langle\chi_{\mathrm{ac}},\lambda\rangle > 0. \end{cases}$$

We remark that as *both* m and $w \in W_m$ vary, one still obtains only *finitely many* cones $\mathscr{C}_w$; they are essentially the various intersections of $\mathscr{C}_1, \ldots, \mathscr{C}_z$ defined above, corresponding to the *fixed* representation $G \to \mathrm{Gl}(V)$. We denote them by $\mathscr{C}'_1, \ldots, \mathscr{C}'_Z$.

Proposition 2.1. *For $m \gg 0$ holds:*

$$\big(E\times V^m\big)^{\mathrm{s}}(K^\times \times G, (r+1)\chi_t + m\chi_{\mathrm{ac}}) = \big(E\setminus\{0\}\big) \times \big(V^m\big)^{\mathrm{s}}(G,\chi_{\mathrm{ac}}).$$

Observe that the statement is similar in spirit to [8, Proposition 2.18], with the difference that here we vary the basis as well.

Proof. We start proving the '$\supset$' inclusion. Let $w = (\varphi,\underline{v}) \in (E\setminus\{0\}) \times (V^m)^{\mathrm{s}}(G,\chi_{\mathrm{ac}})$. By definition, this means that

$$m(\underline{v},\lambda) \geqslant 0 \quad \Rightarrow \quad \langle\chi_{\mathrm{ac}},\lambda\rangle > 0.$$

The conditions (A) and (B) in (2.1) are automatically fulfilled. We wish to prove that for large m the condition (C) holds too. Let λ_0 be such that $m(\varphi,\lambda_0) \geqslant 1$ and $m(\underline{v},\lambda_0) \geqslant 0$.

We recall that only finitely many cones $\mathscr{C}'_w$ appear when m and w vary. On each such cone, the linear function $\langle \chi_{\mathrm{ac}}, \cdot \rangle$ is strictly positive, and we find $c_1 > 0$ such that $\langle \chi_{\mathrm{ac}}, \lambda \rangle \geqslant c_1|\lambda|$ for all $\lambda \in \mathscr{C}'_1 \cup \ldots \cup \mathscr{C}'_Z$.

For fixed φ, the function $m(\varphi, \cdot)$ is piecewise linear. Furthermore, as φ varies, $m(\varphi, \cdot)$ depends only on the weights of T on M. Overall we find a constant $c_2 > 0$ *independent of* φ such that $|m(\varphi, \lambda)| \leqslant c_2|\lambda|$ for all $\lambda \in \mathcal{X}_*(T)$.

Back to our case, we obtain

$$c_2|\lambda_0| \geqslant m(\varphi, \lambda_0) \geqslant 1 \quad \Rightarrow \quad |\lambda_0| \geqslant \frac{1}{c_2},$$

and deduce

$$-(r+1) + m \cdot \langle \chi_{\mathrm{ac}}, \lambda_0 \rangle \geqslant -(r+1) + m \cdot c_1|\lambda_0| \geqslant -(r+1) + m \cdot \frac{c_1}{c_2}.$$

It follows that for $m > \frac{(r+1)c_2}{c_1}$ the condition (C) is satisfied.

Conversely, we wish to prove the '$\subset$' inclusion. A quick proof goes as follows: the invariant quotient $E \times V^m /\!\!/ K^\times \times G$ contains $\mathbb{P}(\mathcal{E})$ as an open subset. Since this latter is projective, we deduce the equality.

We give now a proof for this inclusion in the same spirit as before. Consider $(\varphi, \underline{v}) \in \big(E \times V^m\big)^{\mathrm{s}}(K^\times \times G, (r+1)\chi_t + m\chi_{\mathrm{ac}})$, and let $\lambda \in \mathcal{X}_*(G)$ such that $m(\underline{v}, \lambda) \geqslant 0$. We wish to prove that $\langle \chi_{\mathrm{ac}}, \lambda \rangle > 0$. First we observe that the stability w.r.t. $K^\times$ forces $\varphi \neq 0$. We distinguish two cases:

Case 1: If $m(\varphi, \lambda) \geqslant 0$, the conclusion follows from condition (A).

Case 2: If $m(\varphi, \lambda) < 0$, we multiply λ be a positive constant, and may assume that $m(\varphi, \lambda) = -1$. The inequality $\langle \chi_{\mathrm{ac}}, \lambda \rangle > 0$ follows from (B). Here we should remark that this inequality is invariant under multiplication of λ by a positive constant. □

3. The main result

In this section we are going to prove the main result of the paper. We define the *anti-canonical character* of the G-module V to be the character of the G-action on $\det V$. Explicitly, it is

$$\chi_{\mathrm{ac}}(G, V) := \sum_{\omega \in \mathcal{X}} \nu_\omega d_\omega \chi_\omega \in \mathcal{X}^*(G).$$

I am going to write χ_{ac} for shorthand.

Lemma 1.2 implies that χ_{ac} is an effective character as soon as $\nu_\omega \geqslant d_\omega$ for all $\omega \in \mathcal{X}$. We assume moreover that G acts freely on the corresponding semi-stable locus. It follows that $Y := \mathbb{V}/\!\!/_{\chi_{\mathrm{ac}}} G = \mathbb{V}^{\mathrm{ss}}(G, \chi_{\mathrm{ac}})/G$ is a projective Fano variety.

For $\omega \in \mathcal{X}$, we denote $\mathcal{M}_\omega$ the vector bundle over Y associated to the G-module M_ω. More precisely, $\mathcal{M}_\omega$ corresponds to the module of covariants $\big(K[\mathbb{V}] \otimes_K M_\omega\big)^G$. Observe that the first Chern class of $\mathcal{M}_\omega$ equals $d_\omega \chi_\omega$.

Let me assume now that the characters χ_ω are pairwise distinct. Then the strong Lefschetz property implies that we can find a polarization $\theta \in \mathrm{Pic}(Y)^{\mathrm{ample}} \subset \mathcal{X}^*(G)$ such that the pairings $\langle \chi_\omega \cdot \theta^{\dim Y-1}, [Y]\rangle$ are pairwise distinct. Consequently the slopes of the vector bundles $\mathcal{M}_\omega$ are pairwise distinct, as

$$\mu_\theta(\mathcal{M}_\omega) = \frac{\deg_\theta \mathcal{M}_\omega}{d_\omega} = \langle \chi_\omega \cdot \theta^{\dim Y-1}, [Y]\rangle.$$

Definition 3.1. We fix a polarization θ as before, and define the order $<_\theta$ on $\mathcal{X}$: we declare that $\omega <_\theta \eta$ if $\mu_\theta(\mathcal{M}_\omega) < \mu_\theta(\mathcal{M}_\eta)$.

Proposition 3.2. *Assume that $\nu_\omega \geqslant d_\omega$ for all $\omega \in \mathcal{X}$, and moreover that the characters χ_ω are pairwise distinct. We choose a polarization θ with the property that the slopes $\mu_\theta(\mathcal{M}_\omega)$ are pairwise distinct. Then holds:*

(1) $H^0\big(Y, \mathrm{End}(\mathcal{M}_\omega)\big) = K$ *for all* ω;
(2) $H^0\big(Y, \mathrm{Hom}(\mathcal{M}_\omega, \mathcal{M}_\eta)\big) = 0$ *for* $\omega <_\theta \eta$.

Proof. (i) A section $s \in H^0\big(Y, \mathrm{End}(\mathcal{M}_\omega)\big)$ determines a G-equivariant morphism $S\colon \mathbb{V} \to \mathrm{End}(M_\omega)$ and *vice versa* (the action on $\mathrm{End}(M_\omega)$ is by conjugation). We are going to prove that such a morphism is necessarily a constant.

Since S is G-equivariant, the endomorphism $S_0 \in \mathrm{End}(M_\omega)$ is Ad_G-invariant. By Schur's lemma, $S_0 = c \cdot \mathbb{1}_{M_\omega}$.

Our hypothesis is that $K[\mathbb{V}]^T = K$, and we have proved in lemma 1.1 that there is a 1-PS $\lambda \in \mathcal{X}_*(T)$ such that all its weights on V are strictly positive. In particular $\lim\limits_{t\to 0} \lambda(t)y = 0$ for all $y \in \mathbb{V}$. The G-equivariance implies that

$$S_{\lambda(t)y} = \mathrm{Ad}_{\lambda(t)} \circ S_y \quad\Longrightarrow\quad \lim_{t\to 0} \mathrm{Ad}_{\lambda(t)} \circ S_y = S_0 = c\mathbb{1}_{M_\omega}.$$

The $\lambda(t)$-action on M_ω can be diagonalized in an appropriate basis formed by weight vectors. With respect to these weight spaces S_y has the following block-matrix shape:

$$S_y = \left(\begin{array}{c|c|c} c\mathbb{1} & * & * \\ \hline 0 & c\mathbb{1} & * \\ \hline 0 & 0 & c\mathbb{1} \end{array}\right) \text{ or equivalently } S_y - c\mathbb{1} = \left(\begin{array}{c|c|c} 0 & * & * \\ \hline 0 & 0 & * \\ \hline 0 & 0 & 0 \end{array}\right),\ \forall\, y \in \mathbb{V}.$$

We denote by $\mathfrak{N}_\lambda$ the nilpotent Lie algebra formed by matrices having this shape. Intrinsically, $\mathfrak{N}_\lambda = \{A \in \mathrm{End}(M_\omega) \mid \lim\limits_{t\to 0} Ad_{\lambda(t)} \circ A = 0\}$. We denote

$$\mathrm{Ker}(\mathfrak{N}_\lambda) := \bigcap_{N\in\mathfrak{N}_\lambda} \mathrm{Ker}(N).$$

It is a non-zero subvector space of M_ω.

Applying again the G-equivariance, we deduce that for any $g \in G$ holds:

$$Ad_{g^{-1}} \circ \big(S_y - c\mathbb{1}\big) = S_{g^{-1}y} - c\mathbb{1} \in \mathfrak{N}_\lambda.$$

It follows that

$$\mathrm{Ker}\big(S_y - c\mathbb{1}\big) \supset g\cdot\mathrm{Ker}(\mathfrak{N}_\lambda), \quad \forall\, g \in G \implies \mathrm{Ker}\big(S_y - c\mathbb{1}\big) \supset \sum_{g\in G} g\cdot\mathrm{Ker}(\mathfrak{N}_\lambda).$$

This latter is a non-zero G-sub-module of M_ω. Since M_ω is irreducible we deduce $\mathrm{Ker}\big(S_y - c\mathbb{1}\big) = M_\omega$, and therefore $S_y = c\mathbb{1}$.

(ii) This is an immediate application of the stability result proved in the previous section. According to theorem 1.3, $\mathcal{M}_\omega \to Y$ are semi-stable with respect to any polarization, as soon as $\nu_\omega \geqslant d_\omega$ for all ω. The slope semi-stability property implies that $H^0\big(\mathrm{Hom}(\mathcal{M}_\omega, \mathcal{M}_\eta)\big) = 0$ for $\mu_\theta(\mathcal{M}_\omega) < \mu_\theta(\mathcal{M}_\eta)$. □

Proposition 3.3. *Assume that $\nu_\omega \gg d_\omega$ for all $\omega \in \mathcal{X}$. Then*

$$H^i\big(Y, \mathrm{Hom}(\mathcal{M}_\omega, \mathcal{M}_\eta)\big) = 0, \quad \textit{for } \omega \neq \eta \textit{ and } i > 0.$$

Proof. Let $\omega, \eta \in \mathcal{X}$ and define $E := \mathrm{End}(M_\omega \oplus M_\eta)$; the associated vector bundle on Y is

$$\begin{aligned}\mathcal{E} = \mathrm{End}(\mathcal{M}_\omega \oplus \mathcal{M}_\eta) = \mathrm{End}(\mathcal{M}_\omega) &\oplus \mathrm{Hom}(\mathcal{M}_\omega, \mathcal{M}_\eta)\\ &\oplus \mathrm{Hom}(\mathcal{M}_\eta, \mathcal{M}_\omega) \oplus \mathrm{End}(\mathcal{M}_\eta).\end{aligned}$$

We use the relative duality for the fibration $\mathbb{P}(\mathcal{E}^\vee) \xrightarrow{\pi} Y$ to reduce the vanishing of the higher cohomology of E over Y to the cohomology vanishing of a certain negative bundle over $\mathbb{P}(\mathcal{E}^\vee)$. Denote $r = r_{\omega,\eta} := \dim E = (d_\omega + d_\eta)^2$.

$$\begin{aligned} H^{n-i}(Y, \mathcal{E}) &\cong H^{n-i}\big(\mathbb{P}(\mathcal{E}^\vee), \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1)\big)\\ &\cong H^{n-i}\big(\mathbb{P}(\mathcal{E}^\vee), K_{\mathbb{P}(\mathcal{E}^\vee)} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)\big)^\vee\\ &= H^{(r-1)+i}\big(\mathbb{P}(\mathcal{E}^\vee), \pi^* K_Y \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-r-1)\big)^\vee. \end{aligned}$$

The Kodaira vanishing theorem implies that $H^j(Y, \mathcal{E})$ vanishes for $j = 1, \ldots, n$ as soon as $\pi^* K_Y^{-1} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(r+1)$ is a positive (that is ample) line bundle over $\mathbb{P}(\mathcal{E}^\vee)$.

Consider the $K^\times \times G$-module $E \oplus V$, where the module structure is given by

$$(t, g) \times (h, v) := \big(t \cdot \rho(g) h \rho(g)^{-1}, \rho(g) v\big),$$

and denote χ_t the projection $K^\times \times G \to K^\times$. The condition above can be rephrased by saying that $\mathbb{P}(\mathcal{E}^\vee)$ is the invariant quotient of $\mathbb{E} \times \mathbb{V}$ by $K^\times \times G$, corresponding to the character $(r+1)\chi_t + \chi_{\rm ac}$. This is proved in proposition 2.1. $\square$

Combining the propositions 3.2 and 3.3, we obtain the main result, announced in the introduction.

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